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Functional Dependence of Lagrange Multipliers*

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ABSTRACT

Given three variational problems consisting of minimizing a given function(al) subject to one constraint function(al), the three equivalent Lagrange-multiplier problems are deduced and the proper functional dependence of the particular Lagrange multipliers is derived. The results prove the consistency of the use and subsequent physical interpretation of the constant Lagrange multipliers in the plasma, global-stability, variational calculations with integral constraints of Chandrasekhar, of Woltzer, and of Wells and Norwood.

I. INTRODUCTION

One of the more subtle questions in the use of Lagrange-multiplier techniques[1] is the physical interpretation[2] and explicit determination[3] of particular multipliers without ad hoc assumptions[4]. Even more basic than these queries is the question of functional dependence; viz., when _must_ Lagrange multipliers in a given variational problem be _constant_ and when can they be spatially dependent? A perusal of the standard literature[1] shows cases of both constant and spatially-dependent[4] Lagrange multipliers, but seemingly no sharp, necessary and sufficient conditions of distinction between these two cases.

In the current studies of Wells and Norwood[5] on global stability of closed plasma configurations, the functional dependence of certain of the multipliers is crucial.

When one treads on the essentially unfamiliar ground of constraint conditions on minimum-entropy production rates, such as typified by the recent work of Robertson[6], _any_ guiding conditions on the Lagrange multipliers would seem to help considerably.

In section II three given variational problems consisting of minimizing a function(al) subject to one constraint function(al) are reset in the Lagrange-multiplier formalism.

In section III the functional dependence of the Lagrange-multiplier of problem 1 is first analyzed by the constraint-elimination method and then shown to be consistent with a similar analysis in the Lagrange-multiplier formalism.

Section IV and V contain similiar analyses for problems 2 and 3.

II. THREE VARIATIONAL PROBLEMS

For simplicity[7], we restrict the discussion below to the scalar function(als) f; F, J

$$f = f(\underset{\sim}{x}) \quad , \tag{1}$$

$$F[x^1] = \int_{t_0}^{t_1} \tilde{F}\,(t,x^1,\dot{x}^1)\,dt \quad , \tag{2a}$$

$$J[\underset{\sim}{x}] = \int_{t_0}^{t_1} \tilde{J}(t,\underset{\sim}{x},\dot{\underset{\sim}{x}})dt \quad , \tag{2b}$$

and one constraint function(al) g; G

$$g(\underset{\sim}{x}) \equiv k_1 \quad , \tag{3}$$

$$G[x^1] \equiv \int_{t_0}^{t_1} \tilde{G}(t,x^1,\dot{x}^1)dt = k_2 \quad , \tag{4}$$

where $\underset{\sim}{x} \equiv x^i\hat{e}_i$; $(i=1,2)$, $\dot{\underset{\sim}{x}} \equiv d(\underset{\sim}{x})/dt$, $\underset{\sim}{\nabla} \equiv \hat{\underset{\sim}{e}}^i\partial_i$.

The following three variational problems are formulated:

$$1'.)\ \text{STN. } f(\underset{\sim}{x}) \ni g(\underset{\sim}{x}) = k_1 \ , \tag{4}$$

$$2'.)\ \text{STN. } J[\underset{\sim}{x}] \ni g(\underset{\sim}{x}) = k_1 \ , \tag{5}$$

$$3'.)\ \text{STN. } F[x^1] \ni G[x^1] = k_2 \ , \tag{6}$$

using the notation (STN.) for stationarity of the given function(al). It is easy to show that problems 1′-3′ can be replaced in the Lagrange-multiplier formalism by

$$1.)\ \text{STN.}\{H_1(\underset{\sim}{x},\lambda_1) \equiv f + \lambda_1(g-k_1)\} \ , \tag{7}$$

$$2.)\ \text{STN.}\int_{t_0}^{t_1} dt\{\tilde{J} + \lambda_2(\underset{\sim}{x})\ dg\} \quad , \tag{8}$$

$$3.)\ \text{STN. } \{H_3[x^1,\lambda_3] \equiv F + \lambda_3\ (G-k_2)\} \quad , \tag{9}$$

respectively, where now the unprimed problems are made stationary as a function of the given variables (assumed independent), subject to no constraints.

At this point all of the above noted sources[1-4] work out particular examples of <u>problems 1-3</u> in which λ_1 and λ_3 are constant but occasionally[4] permit λ_2 to be nonconstant.

The justification for making λ_3 constant usually[1] is left to the unsatisfactory question: How else can one perform the variation without using the constancy of λ_3 to commute with the integral? This can be easily repudiated by postulating that λ_3 is really $\lambda_3 \equiv \langle\tilde{\lambda}_3\rangle$ and that λ_3 could be taken inside to $\tilde{\lambda}_3$.

Even more difficult to intuit is the constancy of λ_1 in the light of <u>problem 2</u>.

The main purpose of this note is to show that the Lagrange multiplier formalism <u>explicitly</u> <u>requires</u> λ_1 and λ_3 to be constants but permits spatial variation of λ_2. Note that this result justifies the integral-constraint technique and the consistency of the density as a <u>constant</u> Lagrange multiplier in the incompressible-plasma model of Wells[5]. This interpretation of the Lagrange multiplier subsequently[5] forced a modification of one of the constraint integrals for the compressible plasma model.

III. PROBLEMS 1′ AND 1

It should of course be realized that any constrained variational problem can, in principle, be solved by explicit elimination of the constraint function(al) without recourse to the Lagrange-multiplier technique. In this context, the Lagrange-multiplier formalism is a mathematical technique which is equivalent to, but sometimes much more convenient than, the alternate constraint-elimination method. However, this sometimes ease of calculation is paid for with the added duty of analyzing and physically interpreting the multipliers. These problems were of course discussed briefly in section I.

It is easy to show that the constraint-elimination solution method applied to problem 1′ reduces the necessary[8] conditions of solution to

$$\underset{\sim}{\nabla} f \times \underset{\sim}{\nabla} g = 0 \quad , \tag{10a}$$

$$g = k_1 \quad , \tag{10b}$$

where the cross-product in Eq.(10a) is in the x^3 direction.

Equation (10a) implies that

$$\underset{\sim}{\nabla} f = -\tilde{\lambda}_1(\underset{\sim}{x})\underset{\sim}{\nabla} g \quad , \tag{11}$$

where $\tilde{\lambda}_1$ is an arbitrary scalar function.

To determine $\tilde{\lambda}_1$ we examine three cases:

CASE 1. $f_1 = f_1(g)$, (f_1 a total function of g). (12)

By calculating the gradient of Eq. (12a), we obtain

$$\underset{\sim}{\nabla} f_1 = (df_1/dg)\underset{\sim}{\nabla} g \quad . \tag{13}$$

Since by assumption we have f_1 as a total function of g,

$$df_1/dg = \phi(g), \tag{14}$$

$$= -\tilde{\lambda}_1 \text{ (constant along } g = k_1). \tag{15}$$

This case of course gives the trivial result that f is stationary at every point along $g = k_1$, but was included for completeness.

CASE 2: $f_2 \neq f_2(g)$ (f_2 not a total function of g). (12b)

A simple sketch of either f_2 evaluated along $g_2 = k_1$ or lines of $f_2 \equiv$ constant relative to $g \equiv k_1$ shows that Eq. (11) can be satisfied at only a discrete number of points. By evaluating $\tilde{\lambda}_1$ at these points, we obtain a set of constants which can be considered an equivalent set of constant Lagrange multipliers.

Hence for both cases the constraint-elimination method explicitly reduces the Lagrange multipliers to constants.

CASE 3: case 1 for part of domain of $g = k_1$ and case 2 for remainder. (12c)

Results of case 1 and case 2 follow directly, requiring again the constancy of the Lagrange multiplier $\tilde{\lambda}_1$.

Now as a consistency check, we analyze problem 1 initially assuming $\lambda_1 \equiv \lambda_1(\underset{\sim}{x})$.

The necessary[8] conditions for the H_1 of problem 1 to be stationary are

$$\underset{\sim}{\nabla} f + \lambda_1 \underset{\sim}{\nabla} g = -g \underset{\sim}{\nabla} \lambda_1 \quad , \tag{16a}$$

$$g = k_1 \quad . \tag{16b}$$

Computing [Eq. (16a)] $\times \underset{\sim}{\nabla} g$ gives

$$\underset{\sim}{\nabla} f \times \underset{\sim}{\nabla} g = -g \underset{\sim}{\nabla} \lambda_1 \times \underset{\sim}{\nabla} g \quad , \tag{16c}$$

which implies $\underset{\sim}{\nabla} g \sim \underset{\sim}{\nabla}\lambda_1$ to be consistent with Eq. (10a). However the calculation of $(\underset{\sim}{\nabla} g)\cdot$ [Eq. (16a)] shows that the only functional dependence of λ_1 permitted for consistency with the constraint-elimination solution of problem 1´ is that of $\lambda_1 \equiv$ constant.

Both methods are therefore consistent and require $\lambda_1 \equiv$ constant.

IV. PROBLEM 3

Problem 3 is treated next as it also implies a constant λ_3.

Note first the usual necessary[1,10] condition for the stationarity of a functional:

$$\delta F \equiv \epsilon_1 \left\{ d(F[x_1 + \epsilon_1 \eta_1]) \Big/ d\epsilon_1 \Big|_{\epsilon_1 = 0} \right\} = 0 , \tag{17}$$

where the δ operator defines the first variation of the functional F in the usual notation.

We now evaluate the functionals $F[x^1]$ and $G[x^1]$ at the point $x'^1 \equiv x^1 + \epsilon_1 \eta_1 + \epsilon_2 \eta_2$:

$$F[x'^1] \equiv F[\epsilon_1, \epsilon_2] \quad , \tag{18}$$

$$\geq F[0,0] \quad , \tag{19}$$

where

$$F[0,0] = F[x_1] \quad , \tag{20}$$

and

$$G[x'^1] \equiv G[\epsilon_1, \epsilon_2] \quad , \tag{21}$$

$$\equiv G[0,0] \quad , \tag{22}$$

$$\equiv G[x_1] = k_2 \quad , \tag{23}$$

where Eqs. (19) and (23) follow from the definitions of functional stationarity and functional constraint, respectively.

Substituting Eqs. (18) and (21) into Eq. (9) and making stationary the result with respect to $\epsilon_1, \epsilon_2, \lambda_3$ by using the δ variation from Eq. (17), we can reduce problem 3 to the form:

$$\{{}^{\partial F}/\partial\epsilon_1 + \lambda_3\ {}^{\partial G}/\partial\epsilon_1\}\Big|_{\epsilon_1=0} = 0 \quad , \tag{24a}$$

$$\{{}^{\partial F}/\partial\epsilon_2 + \lambda_3\ {}^{\partial G}/\partial\epsilon_2\}\Big|_{\epsilon_2=0} = 0 \quad , \tag{24b}$$

$$\{G = k_2\}\Big|_{\epsilon_1,\epsilon_2=0} \quad . \tag{24c}$$

The equivalence of Eq. (24) and the Eqs. (16) which define problem 1 proves by inspection the iff conditions for constancy of the λ_3 Lagrange multiplier. This constancy of λ_3 permits its commutation with the integration and the usual reduction to the Euler-Lagrange[1] operator $[\,]_x \equiv d/dt\,(d/d\dot{x}) - d/dx$ acting on the function $\tilde{F} + \lambda_3\tilde{G}$. Also note the consistency restriction on G of $[G]_x \neq 0$. A physical interpretation of this restriction seems lacking in the literature.

V. PROBLEM 2

Using the Euler-Lagrange operator $[\,]_{x^i}$ and the variations $\tilde{x}^i = x^i + \epsilon^i \eta^i$, where i = 1,2 and with no summation over repeated indices, we can easily reduce Eq. (8) of problem 2 to

$$[\tilde{J}]_{x^i} + \lambda_2(\underset{\sim}{x})\, {}^{\partial g}/_{\partial x^i} = 0 \quad , \tag{25a}$$

$$g = k_1 \quad . \tag{25b}$$

Inspection of Eq. (25) shows that the possible nonconstancy of λ_2 results from noting either

1.) the lack of constraints analogous to those of problem 1 which require λ_2 to be constant.

or

2.) the integral of g over the interval is equivalent to an infinite number of point constraints, each of which has a constant Lagrange multiplier but which in the limit of the Riemann sum produces a $\lambda_2(\underset{\sim}{x})$.

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3 Note the nontrivial calculations needed to prove even the positive-definiteness of the viscosity coefficients of fluid mechanics: L.D. Landau and E.M. Lifshitz, Fluid Mechanics (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1959), p. 54; note the extensive vector calculations needed to determine and eliminate the Lagrange multipliers from even simple variational calculations: T.S. Lundgren, Phys. Fluids. 6, 393 (1963).

[4] E.g., notice the ad hoc determination of the Lagrange multipliers in the variational calculation of P. Rosen, Phys. Fluids. 1, 251 (1953).

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[6] H.S. Robertson, Phys. Rev. 188, 233 (1969).

[7] In addition, numerous points of mathematical rigor and finery, such as using different symbols to distinguish between a function and its functional value and between the zero vector and the number zero, etc., have been omitted whenever context permits.

[8] Sufficiency conditions here, of course, entail considerations of the sign and form of the second derivatives; i.e., the Hessian.

[9] See. e.g., S. Chandrasekhar, Proc. Natn. Acad. Sci. (U.S.A.) 42, 5 (1956), Eqs. (17) and (18).

[10] As is well known, the problem of sufficiency conditions is one of the more difficult subjects in the calculus of variations and is beyond the scope of this note. Moreover while Weierstrasz' Theorem insures the existence of the maximum-minimum problem of functions, no such comparably general results exist for the maximum-minimum problem of functionals in the calculus of variations. For a rigorous treatment and extensive bibliography of the results that existed through 1934, see M. Morse, The Calculus of Variations in the Large (American Mathematics Society, New York, 1934), Chapters I-IV. For a more modern treatment see C.B. Morrey, Jr., Multiple Integrals in the Calculus of Variations (Springer-Verlag Inc., New York, 1966).

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